9.4 - 9.8 OVERVIEW (TIDBITS)

• **Direct Comparison Test**: Let $0 \le a_n \le b_n$

If $\sum_{n=1}^{\infty} b_n$ converges $\Rightarrow \sum_{n=1}^{\infty} a_n$ converges &

If $\sum_{n=1}^{\infty} a_n$ diverges $\Rightarrow \sum_{n=1}^{\infty} b_n$ diverges

Ex: $0 < \underbrace{\dfrac{2^n}{3^n(n+1)}}_{a_n} < \underbrace{\left(\dfrac{2}{3}\right)^n}_{b_n}$

Since $\sum_{n=1}^{\infty} (2/3)^n$ converges, $\sum_{n=1}^{\infty} 2^n/3^n(n+1)$ also converges

• **Limit Comparison Test**: Given $\sum_{n=K}^{\infty} a_n$ & $\sum_{n=K}^{\infty} b_n$ where $a_n \ge 0$, $b_n > 0$ for all n.

If $\lim_{n\to\infty} \dfrac{a_n}{b_n}$ is positive & finite, then either both series converge or both series diverge.

Ex: $S = \sum_{n=1}^{\infty} \dfrac{n^2+n}{n^3-2n+2}$. Choose $\sum_{n=1}^{\infty} \dfrac{1}{n}$ ($n^2/n^3 = 1/n$)

$\lim_{n\to\infty} \left(\dfrac{n^2+n}{n^3-2n+2}\right) \div \left(\dfrac{1}{n}\right) = \lim_{n\to\infty} \left(\dfrac{n^2+n}{n^3-2n+2}\right)\cdot\left(\dfrac{n}{1}\right)$

$\lim_{n\to\infty} \dfrac{n^3+n^2}{n^3-2n+2} \cdot \dfrac{1/n^3}{1/n^3}$

$\lim_{n\to\infty} \dfrac{1+1/n}{1-2/n^2+2/n^3} = \dfrac{1+0}{1-0+0} = 1 \leftarrow$ Finite & positive.

Since $\sum_{n=1}^{\infty} (1/n)^1$ diverges, $\sum_{n=1}^{\infty} n^2+n/n^3-2n+2$ diverges.

• **Alternating Series Test**: Given $\sum_{n=K}^{\infty} a_n$, & $a_n = (-1)^n \cdot b_n$ or $a_n = (-1)^{n+1} \cdot b_n$.

If ① $\lim_{n\to\infty} b_n = 0$ AND ② $\{b_n\}$ is a dec. sequence, then $\sum_{n=K}^{\infty} a_n$ converges.

Ex: $\sum_{n=1}^{\infty} \dfrac{(-1)^{n+1}}{n^2} = \sum_{n=1}^{\infty} (-1)^{n+1} \underbrace{\left(\dfrac{1}{n}\right)^2}_{b_n}$

① $\lim_{n\to\infty} (1/n)^2 = 0$ ✓ & ② $(1/n)^2$ is a dec. sequence

$\Rightarrow \sum_{n=1}^{\infty} (-1)^{n+1}/n^2$ converges.

• **Ratio Test**: Given $\sum_{n=K}^{\infty} a_n$ & $\lim_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| = L$

Ex: $\sum_{n=1}^{\infty} \underbrace{\dfrac{n+2}{n!}}_{a_n}$ $\lim_{n\to\infty} \left|\dfrac{n+3}{(n+1)!} \cdot \dfrac{n!}{n+2}\right|$

$\lim_{n\to\infty} \left|\dfrac{n+3}{(n+1)\cdot \cancel{n!}} \cdot \dfrac{\cancel{n!}}{n+2}\right| = 0 < 1$ Series is abs. conv.

• If $L < 1$, series is absolutely convergent

• If $L > 1$, series is divergent

• If $L = 1$, test is inconclusive

• **Conditional / Abs. convergence**: If $\sum_{n=k}^{\infty} a_n$ converges & $\sum_{n=k}^{\infty} |a_n|$ diverges, series is conditionally conv.

If $\sum_{n=k}^{\infty} a_n$ converges & $\sum_{n=k}^{\infty} |a_n|$ converges, series is absolutely convergent.

Ex: $\sum_{n=2}^{\infty} (-1)^n \cdot \overbrace{\frac{\sqrt{n}+1}{2n-3}}^{b_n}$	$\sum_{n=2}^{\infty} \overbrace{\left\lvert (-1)^n \cdot \frac{\sqrt{n}+1}{2n-3} \right\rvert}^{a_n} = \sum_{n=2}^{\infty} \frac{\sqrt{n}+1}{2n-3}$
Alt. Series Test: ① $\lim_{n\to\infty} b_n = 0$ ✓	$\sum_{n=2}^{\infty} \frac{\sqrt{n}+1}{2n-3} \div \frac{1}{n^{1/2}} = \sum_{n=2}^{\infty} \frac{\sqrt{n}+1}{2n-3} \cdot \frac{\sqrt{n}}{1} = \frac{1}{2}$
② $\{b_n\}$ is a dec. seq. ✓	$1/2$ is both pos. & finite, since
∴ $\sum_{n=2}^{\infty} (-1)^n \cdot \frac{\sqrt{n}+1}{2n-3}$ converges	$\sum_{n=2}^{\infty} \left(\frac{1}{n}\right)^{1/2}$ diverges (p-series), $\sum_{n=2}^{\infty} a_n$ div. (Limit comparison Test).

Hence $\sum_{n=2}^{\infty} (-1)^n \cdot \frac{\sqrt{n}+1}{2n-3}$ converges conditionally.

Ex: $\sum_{n=1}^{\infty} (-1)^{n+1} \cdot \overbrace{\frac{2^n}{5\cdot 3^{n+1}}}^{b_n}$	$\sum_{n=1}^{\infty} \lvert a_n \rvert = \sum_{n=1}^{\infty} {}^{2^n}/_{5\cdot 3^{n+1}}$
① $\lim_{n\to\infty} \frac{2^n}{15\cdot 3^n} = 0$ ✓	$= \frac{1}{15} \sum_{n=1}^{\infty} (2/3)^n$, so
② $\{b_n\}$ is a dec. seq. ✓	$\sum_{n=1}^{\infty} \left\lvert (-1)^{n+1} \cdot {}^{2^n}/_{5\cdot 3^{n+1}} \right\rvert$
∴ $\sum_{n=1}^{\infty} (-1)^{n+1} \cdot {}^{2^n}/_{5\cdot 3^{n+1}}$ converg.	converges. (Geometric Series)

Hence $\sum_{n=1}^{\infty} (-1)^{n+1} \cdot {}^{2^n}/_{5\cdot 3^{n+1}}$ converges absolutely.

• **Maclaurin Polynomials**: $\sum_{n=0}^{\infty} \frac{f^n(0)\cdot x^n}{n!} = f(0) + f'(0)\cdot x + \frac{f''(x)\cdot x^2}{2!} + \frac{f^3(x)\cdot x^3}{3!} + \ldots$

(Always centered @ $x=0$)

Ex: Find the Maclaurin Poly of degree 4 for $f(x) = e^{2x}$	$p(x) = f(0) + f'(0)\cdot x + \frac{f''(0)\cdot x^2}{2!} + \frac{f^3(0)\cdot x^3}{3!} + \frac{f^4(0)\cdot x^4}{4!}$

$f(x) = e^{2x} \Rightarrow f(0) = 1$	$p(x) = 1 + 2x + \frac{4x^2}{2!} + \frac{8x^3}{3\cdot 2} + \frac{16x^4}{4\cdot 3\cdot 2}$
$f'(x) = 2e^{2x} \Rightarrow f'(0) = 2$	
$f''(x) = 4e^{2x} \Rightarrow f''(0) = 4$	$p(x) = 1 + 2x + 2x^2 + \frac{4}{3}x^3 + \frac{2}{3}x^4$
$f^3(x) = 8e^{2x} \Rightarrow f^3(0) = 8$	
$f^4(x) = 16e^{2x} \Rightarrow f^4(0) = 16$	You're basically representing $f(x) = e^{2x}$ as a polynomial function. The more terms = the closer the polynomial function looks to e^{2x} centered @ 0.

• **Taylor Polynomials**: $\sum_{n=0}^{\infty} \frac{f^n(a)\cdot(x-a)^n}{n!} = f(a) + f'(a)\cdot(x-a) + \frac{f''(a)\cdot(x-a)^2}{2!} + \frac{f^3(a)\cdot(x-a)^3}{3!} + \ldots$

(centered at $x=a$)

Ex: Find the 3rd Degree Taylor Polynomial, centered @ $x=1$, for $f(x) = 1/x$.	$p(x) = f(1) + f'(1)\cdot(x-1) + \frac{f''(1)\cdot(x-1)^2}{2!} + \frac{f'''(1)\cdot(x-1)^3}{3!}$

$f(x) = 1/x = x^{-1} \Rightarrow f(1) = 1$	$p(x) = 1 + (-1)(x-1) + \frac{2(x-1)^2}{2} + \frac{(-6)\cdot(x-1)^3}{3\cdot 2}$
$f'(x) = -1/x^2 \Rightarrow f'(1) = -1$	
$f''(x) = 2/x^3 \Rightarrow f''(1) = 2$	$p(x) = 1 - (x-1) + (x-1)^2 - (x-1)^3$
$f'''(x) = -6/x^4 \Rightarrow f^3(1) = -6$	

Note: Go on Desmos.com & graph $y = 1/x$.	Then graph $p(x) = 1 - (x-1) + (x-1)^2 - (x-1)^3 + (x-1)^4 - (x-1)^5 + - \ldots$ Pretty Cool!

• **Root Test**: Given $\sum_{n=k}^{\infty} a_n \ni \lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} |a_n|^{1/n} = L$

Ex: $\sum_{n=0}^{\infty} \left(\frac{5n-3n^3}{7n^3+2}\right)^n$ | $\lim_{n\to\infty} \sqrt[n]{\left|\left(\frac{5n-3n^3}{7n^3+2}\right)^n\right|}$

$\lim_{n\to\infty} \left|\frac{5n-3n^3}{7n^3+2}\right| = 3/7 < 1$ | ∴ series is abs. conv.

• If $L < 1$, series is absolutely convergent
• If $L > 1$, series is divergent
• If $L = 1$, test is inconclusive

• **Interval of convergence**: What values of x make the series converge?

Ex: $\sum_{n=1}^{\infty} \frac{(x-4)^n}{n\cdot(-2)^n}$ | $a_{n+1} = \frac{(x-4)^{n+1}}{(n+1)\cdot(-2)^{n+1}}$ | $a_n = \frac{(x-4)^n}{n\cdot(-2)^n}$ | $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$

$\lim_{n\to\infty} \left|\frac{(x-4)^1\cdot\cancel{(x-4)^n}}{(n+1)\cancel{(-2)^n}(-2)} \cdot \frac{n\,\cancel{(-2)^n}}{\cancel{(x-4)^n}}\right| = \lim_{n\to\infty} \left|\frac{(x-4)\,n}{-2(n+1)}\right| = \left|\frac{x-4}{2}\right|$

Series converges when $\left|\frac{x-4}{2}\right| < 1$ by Ratio Test | $-1 < \frac{x-4}{2} < 1$ | $-2 < x-4 < 2$, $2 < x < 6$ | Check endpoints...

$\sum_{n=1}^{\infty} \frac{(2-4)^n}{n\cdot(-2)^n} = \sum_{n=1}^{\infty} \frac{1}{n} \Rightarrow$ diverges (p-series) | $\sum_{n=1}^{\infty} \frac{(6-4)^n}{n(-2)^n} = \sum_{n=1}^{\infty} \frac{\cancel{(2)^n}}{n(-1)^n\cancel{(2)^n}} = \sum_{n=1}^{\infty} (-1)^n\cdot\frac{1}{n}$

Let $b_n = 1/n$ | $\lim_{n\to\infty} b_n = 0$ ✓, $\{b_n\}$ is a dec seq. ✓ | $\sum_{n=1}^{\infty} (-1)^n\cdot\frac{1}{n}$ converges | ∴ Interval of convergence is $2 < x \le 6$.

AP Calculus BC Chapters 9.4 - 9.8 Study Guide

Note: Use the limit comparison test to determine the convergence/divergence of the series. Show the work that leads to your answer.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{3^n - 1}$

2. $\displaystyle\sum_{n=1}^{\infty} \frac{2n+5}{(n-3)(n-2)}$

3. $\displaystyle\sum_{n=1}^{\infty} \frac{6^n - 2}{5^n}$

4. $\displaystyle\sum_{n=1}^{\infty} \frac{4}{\sqrt{n^2+1}}$

5. $\displaystyle\sum_{n=1}^{\infty} \tan\left(\frac{1}{n^2}\right)$

Note: Determine the convergence/divergence of the series. Show the work that leads to your answer, and detail what test you are using.

6. $\displaystyle\frac{4}{7} + \frac{4}{14} + \frac{4}{21} + \frac{4}{28} + \dots$

7. $\displaystyle\frac{1}{101} + \frac{1}{104} + \frac{1}{109} + \frac{1}{116} + \dots$

8. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{2n+3}$

9. $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^{n+1} \cdot n}{n^2 - 3}$

10. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} \cdot \cos\left[\pi(n+1)\right]$

11. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(n+2)!}$

12. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n \cdot n^2}{4n}$

Note: Determine whether the series converges conditionally or absolutely, or diverges. Show the work that leads to your answer:

13. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \cdot 4^{1/n}$

14. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$

15. $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n \cdot \ln(n)}{n!}$

Note: Use the Ratio Test to determine the convergence / divergence of the series. Show the work that leads to your answer.

(16) $\displaystyle\sum_{n=1}^{\infty} \frac{\ln(n)}{5^n}$ (17) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{4^n}$ (18) $\displaystyle\sum_{n=1}^{\infty} n\left(\frac{5}{4}\right)^{2n+1}$

(19) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n+1)^4}{4^n}$ (20) $\displaystyle\sum_{n=1}^{\infty} \frac{7^n}{3^n+1}$

Note: Use the Root Test to determine the convergence/divergence of the series. Show the work that leads to your answer.

(21) $\displaystyle\sum_{n=1}^{\infty} \frac{7^n}{3^n+1}$ (22) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{n+1}{2n+1}\right)^n$ (23) $\displaystyle\sum_{n=1}^{\infty} \frac{-n^n}{3^{1+2n}}$

Note: Find the Maclaurin Polynomial of degree n for the function:	Note: Find the n^{th} Taylor Polynomial centered at $x = c$:
(24) $f(x) = \dfrac{1}{\sqrt{x+1}}$; $n = 3$	(25) $f(x) = \dfrac{1}{x}$, $n = 5$, $c = 1$

Note: Find the interval of convergence of the power series. Be sure to include a check for convergence at the endpoints.

(26) $\displaystyle\sum_{n=1}^{\infty} \frac{(n+6)(x+4)^n}{n \cdot 7^n}$ (27) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{x}{3}\right)^n$

AP Calc BC 9.4 - 9.8 Study Guide Solutions

① $\sum_{n=1}^{\infty} \frac{2^n}{3^n-1}$ | $\lim_{n\to\infty} \frac{\left(2^n/3^n-1\right)}{\left(2^n/3^n\right)}$ | $\lim_{n\to\infty}\left(\frac{\cancel{2^n}}{3^n-1}\right)\left(\frac{3^n}{\cancel{2^n}}\right) = 1$ ✓ | series converges by the Limit Comp. Test w/ the converg. geo. series $\sum_{n=1}^{\infty}(2/3)^n$

② $\sum_{n=1}^{\infty} \frac{2n+5}{(n-3)(n-2)}$ | $\lim_{n\to\infty} \frac{\left(2n+5/(n-3)(n-2)\right)}{\left(1/n\right)}$ | $\lim_{n\to\infty}\left(\frac{2n+5}{(n-3)(n-2)}\right)\left(\frac{n}{1}\right) = 2$ ✓ | series diverges by the LCT w/ the divergent p-series $\sum_{n=1}^{\infty} 1/n$

③ $\sum_{n=1}^{\infty} \frac{6^n-2}{5^n}$ | $\lim_{n\to\infty} \frac{\left(6^n-2/5^n\right)}{\left(6^n/5^n\right)}$ | $\lim_{n\to\infty}\left(\frac{6^n-2}{\cancel{5^n}}\right)\left(\frac{\cancel{5^n}}{6^n}\right) = 1$ ✓ | series diverges by the LCT w/ the divergent geomet. series $\sum_{n=1}^{\infty}(6/5)^n$

④ $\sum_{n=1}^{\infty} \frac{4}{\sqrt{n^2+1}}$ | $\lim_{n\to\infty} \frac{\left(4/\sqrt{n^2+1}\right)}{1/n}$ | $\lim_{n\to\infty}\left(\frac{4}{\sqrt{n^2+1}}\cdot\frac{n}{1}\right) = 4$ ✓ | series diverges by the LCT w/ the divergent harmonic series $\sum_{n=1}^{\infty} 1/n$.

PSST ⑤ $\sum_{n=1}^{\infty} \tan\left(\frac{1}{n^2}\right)$ | $\lim_{n\to\infty} \frac{\tan\left(1/n^2\right)}{1/n^2}$ | $0/0$ L.R.A. | $\lim_{n\to\infty} \frac{\cancel{-2/n^3}\cdot\sec^2\left(1/n^2\right)}{\cancel{-2/n^3}}$ | $1/\cos^2(0)$ $1/1 = 1$ ✓ | series converg. by the LCT w/ the convergent p-series $\sum_{n=1}^{\infty}(1/n)^2$

⑥ $\frac{4}{7}+\frac{4}{14}+\frac{4}{21}+\frac{4}{28}+\dots$ | $\frac{4}{7}\sum_{n=1}^{\infty}\frac{1}{n}$ | series diverges (harmonic series)

⑦ $\frac{1}{101}+\frac{1}{104}+\frac{1}{109}+\frac{1}{116}+\dots$ | $\sum_{n=1}^{\infty}\frac{1}{100+n^2}$ | $\lim_{n\to\infty}\frac{1/100+n^2}{1/n^2} = \lim_{n\to\infty}\frac{1}{100+n^2}\cdot\frac{n^2}{1} = 1$ ✓ | series converg. by LCT w/ conv. p series $\sum_{n=1}^{\infty}(1/n)^2$

⑧ $\sum_{n=1}^{\infty}\frac{(-1)^n}{2n+3}$ | $\sum_{n=1}^{\infty}(-1)^n\cdot\frac{1}{2n+3}$ | Let $b_n = \frac{1}{2n+3}$ | $\lim_{n\to\infty} b_n = 0$; $\{b_n\}$ is a dec. seq. | $\sum_{n=1}^{\infty}\frac{(-1)^n}{2n+3}$ converges by the Alternat. Series Test

⑨ $\sum_{n=2}^{\infty}(-1)^{n+1}\cdot\frac{n}{n^2-3}$ | Let $b_n = \frac{n}{n^2-3}$ | $\lim_{n\to\infty} b_n = 0$; $\{b_n\}$ is a dec. seq. | series converges by the Alternating Series Test.

10 $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} \cdot \cos[\pi(n+1)]$ | $\frac{1}{1}(1) + \frac{1}{4}(-1) + \frac{1}{9}(1) + \frac{1}{16}(-1) + \dots$ | $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \cdot \frac{1}{n^2}$

Let $b_n = 1/n^2$ | $\lim_{n\to\infty} b_n = 0$ ✓ ; $\{b_n\}$ is a dec. seq. ✓ | $\therefore$ series converges by the Alternating Series Test.

11 $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \cdot \frac{1}{(n+2)!}$ | Let $b_n = \frac{1}{(n+2)!}$ | $\lim_{n\to\infty} b_n = 0$ ✓ ; $\{b_n\}$ is a dec. seq. ✓ | $\therefore$ series converges by the Alt. series Test

12 $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n \cdot n^2}{4n}$ | $\displaystyle\frac{1}{4}\sum_{n=1}^{\infty} (-1)^n \cdot n = \pm\infty \neq 0$ | $\therefore$ series diverges by the n^{th} term test.

13 $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \cdot 4^{1/n}$ | $\displaystyle\lim_{n\to\infty} (-1)^{n-1} \cdot 4^{1/n} = \pm 4^0 = \pm 1 \neq 0$ | series diverges by the n^{th} term test.

14 $\displaystyle\sum_{n=1}^{\infty} (-1)^n \cdot \overbrace{\frac{1}{\sqrt{n}}}^{b_n}$ | $\lim_{n\to\infty} b_n = 0$ ✓ ; $\{b_n\}$ is a dec. seq. ✓ | converges by AST | $\displaystyle\sum_{n=1}^{\infty} \left|\frac{(-1)^n}{\sqrt{n}}\right| = \sum_{n=1}^{\infty} \left(\frac{1}{n}\right)^{1/2}$ | diverges (p-series) | $\therefore$ series converg. conditionally

15 $\displaystyle\sum_{n=2}^{\infty} (-1)^n \cdot \overbrace{\frac{\ln(n)}{n!}}^{b_n}$ | $\lim_{n\to\infty} b_n = 0$ ✓ ; $\{b_n\}$ is a dec. seq. ✓ | converges by AST | $\displaystyle\sum_{n=2}^{\infty} \left|(-1)^n \cdot \frac{\ln(n)}{n!}\right| = \sum_{n=2}^{\infty} \frac{\ln(n)}{n!}$

$a_{n+1} = \ln(n+1)/(n+1)!$; $a_n = \ln(n)/n!$ | $\displaystyle\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\ln(n+1)}{(n+1)!} \cdot \frac{n!}{\ln(n)} = \lim_{n\to\infty} \left(\frac{\ln(n+1)}{(n+1)\cancel{(n)!}} \cdot \frac{\cancel{n!}}{\ln(n)}\right)$ | $\frac{\infty}{\infty} \Rightarrow$ L.R.A.

$\displaystyle\lim_{n\to\infty} \frac{1/n+1}{(n+1)(1/n) + \ln(n) \cdot 1}$ | $\displaystyle\lim_{n\to\infty} \frac{1/n+1}{1 + 1/n + \ln(n)}$ | $\displaystyle\frac{1/\infty}{1 + 1/\infty + \ln(\infty)} = \frac{0}{\infty} = 0 < 1$ | converg. by Ratio Test | $\therefore$ series converg. absolutely.

16 $\displaystyle\sum_{n=1}^{\infty} \frac{\ln(n)}{5^n}$ | $a_{n+1} = \ln(n+1)/5^{n+1}$; $a_n = \ln(n)/5^n$ | $\displaystyle\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\ln(n+1)}{5^{n+1}} \cdot \frac{5^n}{\ln(n)}$

$\displaystyle\lim_{n\to\infty} \frac{\ln(n+1) \cdot \cancel{5^n}}{\cancel{5^n} \cdot 5 \cdot \ln(n)}$ | $\frac{\infty}{\infty}$ L.R.A. | $\displaystyle\lim_{n\to\infty} \frac{1/n+1}{5 \cdot 1/n} = \lim_{n\to\infty} \frac{1}{n+1} \cdot \frac{n}{5} = \frac{1}{5} < 1$ | series converges by the ratio test

(17) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{4^n}$ | $a_{n+1} = 1/4^{n+1}$, $a_n = 1/4^n$ | $\displaystyle\lim_{n\to\infty} \frac{1/4^{n+1}}{1/4^n} = \lim_{n\to\infty} \frac{1}{4\cdot \cancel{4^n}}\cdot\frac{\cancel{4^n}}{1} = \frac{1}{4} < 1$ | ∴ series converges by the Ratio Test

(18) $\displaystyle\sum_{n=1}^{\infty} n\left(\frac{5}{4}\right)^{2n+1}$ | $a_{n+1} = (n+1)(5/4)^{2n+3}$, $a_n = n(5/4)^{2n+1}$ | $\displaystyle\lim_{n\to\infty} \frac{(n+1)\cdot(5/4)^{2n+3}}{n(5/4)^{2n+1}} = \lim_{n\to\infty} \frac{(n+1)\cdot(5/4)^2}{n}$

$\displaystyle\lim_{n\to\infty}\left(\frac{25/16\cdot n + 25/16}{n}\right) = \lim_{n\to\infty}\left(\frac{25}{16} + \frac{25/16}{n}\right) = \frac{25}{16} > 1$ | ∴ series diverges by the Ratio Test

(19) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n+1)^4}{4^n}$ | $a_{n+1} = (-1)^{n+2}(n+2)^4/4^{n+1}$, $a_n = (-1)^{n+1}(n+1)^4/4^n$ | $\displaystyle\lim_{n\to\infty} \frac{(-1)^2\cdot\cancel{(-1)^n}(n+2)^4}{4\cdot\cancel{4^n}}\cdot\frac{\cancel{4^n}}{(-1)^1\cancel{(-1)^n}(n+1)^4} = \frac{1}{4} < 1$ | ∴ series converges by Ratio Test

(20) $\displaystyle\sum_{n=1}^{\infty} \frac{7^n}{3^n+1}$ | $a_{n+1} = \dfrac{7^{n+1}}{3^{n+1}+1}$ | $a_n = \dfrac{7^n}{3^n+1}$ | $\displaystyle\lim_{n\to\infty} \frac{7\cdot\cancel{7^n}}{3\cdot 3^n+1}\cdot\frac{3^n+1}{\cancel{7^n}} = \lim_{n\to\infty}\frac{7(3^n+1)}{3(3^n+1/3)}$

$\displaystyle\lim_{n\to\infty} \frac{7(3^n+1)}{3(3^n+1/3)}\cdot\frac{1/3^n}{1/3^n}$ | $\displaystyle\lim_{n\to\infty}\frac{7(1+1/3^n)}{3(1+1/3^{n+1})} = 7/3 > 1$ | ∴ series diverges by the Ratio Test

(21) $\displaystyle\sum_{n=1}^{\infty} \frac{e^{2n}}{n^n}$ | $\displaystyle\lim_{n\to\infty}\sqrt[n]{\left|\frac{e^{2n}}{n^n}\right|}$ | $\displaystyle\lim_{n\to\infty}\frac{e^{2n/n}}{n^{n/n}}$ | $\displaystyle\lim_{n\to\infty}\frac{e^2}{n} = 0 < 1$ | ∴ series converges by the Root Test.

(22) $\displaystyle\sum_{n=1}^{\infty}\left(\frac{n+1}{2n+1}\right)^n$ | $\displaystyle\lim_{n\to\infty}\sqrt[n]{\left|\left(\frac{n+1}{2n+1}\right)^n\right|}$ | $\displaystyle\lim_{n\to\infty}\frac{n+1}{2n+1} = 1/2 < 1$ | ∴ series converges by the Root Test.

(23) $\displaystyle\sum_{n=1}^{\infty}\frac{-n^n}{3^{1+2n}}$ | $\displaystyle\lim_{n\to\infty}\sqrt[n]{\left|\frac{-n^n}{3^{1+2n}}\right|}$ | $\displaystyle\lim_{n\to\infty}\left(\frac{n^n}{3\cdot 3^{2n}}\right)^{1/n}$ | $\displaystyle\lim_{n\to\infty}\left(\frac{n}{3^{1/n}\cdot 3^2}\right) = \infty > 1$ | ∴ series diverges by the Root Test

(24) $f(x) = (x+1)^{-1/2}$; $n = 3$, centered @ $x = 0$

$$\sum_{n=0}^{\infty} \frac{f^n(0)\cdot x^n}{n!} = f(0) + f'(0)\cdot x + \frac{f''(0)\cdot x^2}{2} + \frac{f^3(0)\cdot x^3}{3!}$$

$f(x) = (x+1)^{-1/2} \Rightarrow f(0) = 1$

$f'(x) = -\tfrac{1}{2}(x+1)^{-3/2} \Rightarrow f'(0) = -\tfrac{1}{2}$

$f''(x) = \tfrac{3}{4}(x+1)^{-5/2} \Rightarrow f''(0) = \tfrac{3}{4}$

$f^3(x) = -\tfrac{15}{8}(x+1)^{-7/2} \Rightarrow f^3(0) = -\tfrac{15}{8}$

$$p(x) = 1 + (-\tfrac{1}{2})\cdot x + \frac{\tfrac{3}{4}\cdot x^2}{2} + \frac{(-\tfrac{15}{8})\cdot x^3}{6}$$

$$p(x) = 1 - \frac{1}{2}x + \frac{3}{8}x^2 - \frac{5}{16}x^3$$

(25) $f(x) = x^{-1}$, $n = 5$, centered @ $x = 1$

$$\sum_{n=1}^{\infty} \frac{f^n(a)\cdot(x-a)^n}{n!} = f(a) + \frac{f'(a)\cdot(x-a)^1}{1!} + \ldots + \frac{f^5(a)\cdot(x-a)^5}{5!}$$

$f(x) = x^{-1} \Rightarrow f(1) = 1$

$f'(x) = -x^{-2} \Rightarrow f'(1) = -1$

$f''(x) = 2x^{-3} \Rightarrow f''(1) = 2$

$f^3(x) = -6x^{-4} \Rightarrow f^3(1) = -6$

$f^4(x) = 24x^{-5} \Rightarrow f^4(1) = 24$

$f^5(x) = -120x^{-6} \Rightarrow f^5(1) = -120$

$$p(x) = 1 + (-1)(x-1) + \frac{2(x-1)^2}{2} + \frac{(-6)(x-1)^3}{6} + \frac{24(x-1)^4}{24} + \frac{(-120)(x-1)^5}{120}$$

$$p(x) = 1 - (x-1) + (x-1)^2 - (x-1)^3 + (x-1)^4 - (x-1)^5$$

(26) $$\sum_{n=1}^{\infty} \frac{(n+6)(x+4)^n}{n\cdot 7^n}$$

$$a_{n+1} = \frac{(n+7)(x+4)^{n+1}}{(n+1)\cdot 7^{n+1}} \qquad a_n = \frac{(n+6)(x+4)^n}{n\cdot 7^n}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+7)(x+4)\cdot\cancel{(x+4)^n}}{(n+1)\cdot\cancel{7^n}\cdot 7}\cdot\frac{n\cdot\cancel{7^n}}{(n+6)\cancel{(x+4)^n}}\right| = \lim_{n\to\infty}\left|\frac{(x+4)(n^2+7n)}{7(n+1)(n+6)}\right| = \left|\frac{x+4}{7}\right|$$

Series converges when $\left|\frac{x+4}{7}\right| < 1$; $-1 < \frac{x+4}{7} < 1$; $-7 < x+4 < 7$; $-11 < x < 3$. Check endpoints...

$$\sum_{n=1}^{\infty} \frac{(n+6)(-11+4)^n}{n\cdot 7^n} = \sum_{n=1}^{\infty} \frac{(-1)^n\cdot 7^n(n+6)}{7^n(n)} \Rightarrow \text{diverges by } n^{th} \text{ term test}$$

$$\sum_{n=1}^{\infty} \frac{(n+6)(3+4)^n}{n\cdot 7^n} = \sum_{n=1}^{\infty} \frac{(n+6)\cancel{(7)^n}}{n\cdot\cancel{7^n}}$$

$\lim_{n\to\infty} \frac{n+6}{n} = 1 \neq 0 \Rightarrow$ diverges by n^{th} term test.

$\therefore$ Interval of convergence is $-11 < x < 3$

(27) $$\sum_{n=1}^{\infty}\left(\frac{x}{3}\right)^n$$

This is a geometric series & converges if & only if $|x/3| < 1$

$-1 < x/3 < 1$

$-3 < x < 3$

Interval of convergence is $-3 < x < 3$.