9.1 - 9.3 OVERVIEW (TIDBITS)

- Geometric Series : $\sum_{n=1}^{\infty} A(r)^n$. If $|r|<1$, series converges to $A/1-r$

  If $|r| \geq 1$, series diverges

- n^{th} Term Test : If $\lim_{n \to \infty} a_n \neq 0$, series diverges.

  If $\lim_{n \to \infty} a_n = 0$, test is inconclusive

- Integral Test : If a_n is positive, decreasing, and continuous over an interval then:

  - If $\int_K^{\infty} f(x)\,dx$ converges, then $\sum_{n=K}^{\infty} a_n$ also converges &
  - If $\int_K^{\infty} f(x)\,dx$ diverges, then $\sum_{n=K}^{\infty} a_n$ also diverges

- P-Series : $\sum_{n=1}^{\infty} \left(\frac{1}{n}\right)^p$. If $p>1$, series converges

  If $0<p\leq 1$, series diverges

- Telescoping Series : When the bulk of the "middle terms" cancel out & only the first few terms & last few terms remain.

  - Ex : $\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right) = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \ldots + \left(\frac{1}{n} - \frac{1}{n+1}\right)$

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AP Calculus BC 9.1 - 9.3 Study Guide

Zack Brenneman

Write the first 5 terms of the following sequence given an.

Pg. 604 #6

① $a_n = \frac{2n}{n+3}$

Pg. 604 #5

② $a_n = \sin\left(\frac{n\pi}{2}\right)$

Pg. 604 #8

③ $a_n = (-1)^{n+1} \cdot \left(\frac{2}{n}\right)$

Write an expression for an, the n^{th} term of the following sequence:

Pg. 604 #23

④ 2, 5, 8, 11, . . .

⑤ OMIT

(Made - Up Problem)

⑥ $\frac{3}{1}, \frac{9}{2}, \frac{27}{6}, \frac{81}{24}, \frac{243}{120}, \ldots$

Determine the convergence or divergence of the sequence w/ the following n^{th} term:

Pg. 604 #51

⑦ $a_n = \frac{3n^2 - n + 4}{2n^2 + 1}$

Pg. 604 #49

⑧ $a_n = (-1)^n \cdot \left(\frac{n}{n+1}\right)$

Pg. 604 #71

⑨ $a_n = \frac{\sin(n)}{n}$

Determine the sum of the convergent series. Show how you arrived to your answer:

Pg. 615 #69

⑩ $\sum_{n=0}^{\infty} \frac{4}{2^n}$

Pg. 615 #47

⑪ $3 - 1 + \frac{1}{3} - \frac{1}{9} + \ldots$

⑫ $\sum_{n=1}^{\infty} \frac{1}{n(n+2)}$

Use the Integral Test to determine the convergence or divergence of the following series. All answers must be justified.

Pg. 622 #8

⑬ $\frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \frac{1}{11} + \ldots$

Pg. 622 #21

⑭ $\sum_{n=1}^{\infty} \frac{4n}{2n^2+1}$

⑮ $\sum_{n=1}^{\infty} ne^{-n}$

For the following p-series, determine whether they are convergent or divergent, & explain why:

Pg. 623 # 37

⑯ $1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \ldots$

Pg. 623 # 42

⑰ $\sum_{n=1}^{\infty} \frac{1}{n^{\pi}}$

Pg. 623 # 35

⑱ $\sum_{n=1}^{\infty} \frac{1}{\sqrt[5]{n}}$

For the following series, determine the convergence or divergence, and specify what test you are using: geometric, integral, telescoping, p-series:

(Made Up Example)

⑲ $\sum_{n=1}^{\infty} \frac{n^2}{e^n}$

(Made Up Example)

⑳ $\sum_{n=0}^{\infty} 3 \cdot \left(\frac{5}{9}\right)^n$

Pg. 623 # 41

㉑ $\sum_{n=1}^{\infty} \frac{1}{n^{1.04}}$

(Made Up Example)

㉒ $\sum_{n=1}^{\infty} \frac{2}{n^3} - \frac{2}{n^4}$

(Made Up Example)

㉓ $\sum_{n=1}^{\infty} \frac{3}{n} - \frac{3}{n+1}$

Pg. 622 # 9

㉔ $\frac{\ln 2}{2} + \frac{\ln 3}{3} + \frac{\ln 4}{4} + \ldots$

BC CALCULUS 9.1 - 9.3 Study Guide Solutions

①
$a_n = \frac{2n}{n+3}$
$a_1 = \frac{2(1)}{(1)+3} = \frac{1}{2}$
$\underset{a_1}{1/2}, \underset{a_2}{4/5}, \underset{a_3}{1}, \underset{a_4}{8/7}, \underset{a_5}{5/4}$

②
$a_n = \sin\left(\frac{\pi n}{2}\right)$
$a_1 = \sin(\pi/2) = 1$
$\underset{a_1}{1}, \underset{a_2}{0}, \underset{a_3}{-1}, \underset{a_4}{0}, \underset{a_5}{1}$

③
$a_n = (-1)^{n+1}\cdot\left(\frac{2}{n}\right)$
$a_1 = (-1)^2(2) = 2$
$\underset{a_1}{2}, \underset{a_2}{-1}, \underset{a_3}{2/3}, \underset{a_4}{-1/2}, \underset{a_5}{2/5}$

④
$2, 5, 8, 11, \ldots$ (+3 +3 +3)
$a_n = a_1 + d(n-1)$ (Arithmetic sequence Formula)
$a_n = 2 + 3(n-1) = 2 + 3n - 3$
$a_n = 3n - 1$

⑥
$\frac{3}{1}, \frac{9}{2}, \frac{27}{6}, \frac{81}{24}, \ldots$ (×3 ×3 ×3 on top; ×2 ×3 ×4 below)
$a_n = \frac{3^n}{n!}$
Note: $a_1 = \frac{3^1}{1}$, $a_2 = \frac{3^2}{2\cdot1}$, $a_3 = \frac{3^3}{3\cdot2\cdot1}$

⑦
$a_n = \frac{3n^2 - n + 4}{2n^2+1}$
$\lim\limits_{n\to\infty} \frac{3n^2 \cancel{-n+4}}{2n^2 \cancel{+1}} = \frac{3}{2}$, converges
Note: The TERMS converge to 3/2, not the sum of the terms.

⑧
$a_n = (-1)^n\cdot\left(\frac{n}{n+1}\right)$
$\lim\limits_{n\to\infty} (-1)^n\cdot\frac{n}{n+1} = \lim\limits_{n\to\infty} (-1)^n\cdot 1$
DNE; oscillates between -1 & 1
Diverges

⑨
$a_n = \frac{\sin(n)}{n}$
$\lim\limits_{n\to\infty} \frac{\sin(n)}{n} = \frac{-1 < \# < 1}{\infty} = 0$

⑩
$\sum_{n=0}^{\infty} \frac{4}{2^n}$
$4\sum_{n=0}^{\infty} \frac{1}{2^n}$
$4\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n$
Geometric Series w/ r = 1/2 converges since $0 < |1/2| < 1$
converges to $a_1/(1-r)$

$a_1 = 4\cdot(1/2)^0 = 4$
↑ First Term
$\frac{4}{1-1/2} = \frac{4}{1/2} = \frac{4}{1}\cdot\frac{2}{1} = 8$

⑪ $3 \underset{\times -1/3}{\curvearrowright} -1 \underset{\times -1/3}{\curvearrowright} +1/3 \underset{\times -1/3}{\curvearrowright} -1/9 + \ldots$

$$\sum_{n=0}^{\infty} 3(-1/3)^n$$

Geometric S. w/ $r = -1/3$

converges since $|-1/3| < 1$

$$\frac{3}{1-(-1/3)} = \frac{3}{4/3} = \frac{9}{4}$$

⑫ $$\sum_{n=1}^{\infty} \frac{1}{n(n+2)}$$

$$\frac{A}{n} + \frac{B}{n+2} = \frac{1}{n(n+2)}$$

$A(n+2) + B(n) = 1$

choose $n = -2, 0$

$-2B = 1$

$B = -1/2$

$2A = 1$

$A = 1/2$

$$\frac{1}{n(n+2)} = \frac{1/2}{n} - \frac{1/2}{n+2} = \frac{1}{2n} - \frac{1}{2(n+2)} = \frac{1}{2}\left(\frac{1}{n} - \frac{1}{n+2}\right)$$

$$\sum_{n=1}^{\infty} \frac{1}{n(n+2)} = \frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+2}\right)$$

$$\frac{1}{2}\left[\left(1 - \overset{\times}{\frac{1}{3}}\right) + \left(\frac{1}{2} - \overset{\times}{\frac{1}{4}}\right) + \left(\overset{\times}{\frac{1}{3}} - \overset{\times}{\frac{1}{5}}\right) + \left(\overset{\times}{\frac{1}{4}} - \overset{\times}{\frac{1}{6}}\right) + \left(\overset{\times}{\frac{1}{5}} - \overset{\times}{\frac{1}{7}}\right) + \ldots\right.$$

continues on next line.

$$\ldots + \left(\overset{\times}{\frac{1}{n-3}} - \overset{\times}{\frac{1}{n-1}}\right) + \left(\overset{\times}{\frac{1}{n-2}} - \overset{\times}{\frac{1}{n}}\right) + \left(\overset{\times}{\frac{1}{n-1}} - \frac{1}{n+1}\right) + \left(\overset{\times}{\frac{1}{n}} - \frac{1}{n+2}\right)$$

Sum of all #'s $= S_n = \frac{1}{2}\left[1 + 1/2 - \frac{1}{n+1} - \frac{1}{n+2}\right]$

$$\lim_{n \to \infty} S_n = 1/2\,(1 + 1/2 - 1/\infty - 1/\infty)$$
$$= 1/2\,(3/2 - 0 - 0) = 3/4$$

⑬ $\frac{1}{3} \underset{+2}{\curvearrowright} + \frac{1}{5} \underset{+2}{\curvearrowright} + \frac{1}{7} \underset{+2}{\curvearrowright} + \frac{1}{9} + \ldots$

$a_n = a_1 + d(n-1)$

$a_n = 3 + 2(n-1) = 2n+1$

$$\sum_{n=1}^{\infty} \frac{1}{2n+1}$$

Let $f(x) = \frac{1}{2x+1}$

Note f is pos., dec., & cont. for $x \geq 1$

$$\frac{1}{2}\int_1^{\infty} \frac{2 \cdot 1}{2x+1}\,dx$$

$u = 2x+1$

$du = 2\,dx$

$$\frac{1}{2}\int_1^{\infty} \frac{1}{u}\,du$$

$$\frac{1}{2} \cdot \ln|2x+1|\Big|_1^{\infty}$$

$F(\infty) - F(1)$

$1/2\,[\ln(\infty) - \ln(3)]$

$1/2\,[\infty - \ln(3)] = \infty$

Since $\int_1^{\infty} 1/2x+1\,dx$ diverges, $\sum_{n=1}^{\infty} 1/2n+1$ also diverges by the Integral Test.

⑭ $$\sum_{n=1}^{\infty} \frac{4n}{2n^2+1}$$

Let $f(x) = \frac{4x}{2x^2+1}$

$$f'(x) = \frac{(2x^2+1)(4) - (4x)(4x)}{(2x^2+1)^2} = \frac{-8x^2+4}{(2x^2+1)^2} = \frac{-4(2x^2-1)}{(2x^2+1)^2}$$

Remember:

$$\int \frac{du}{u^2+a^2} = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C$$

C.N.: $x = \pm\sqrt{2}/2$

P.O.I: $x = N/A$

f': $-$ $|$ $+$ $|$ $-$ at $-\sqrt{2}/2$, $\sqrt{2}/2$

$$\frac{1}{2} < \frac{\sqrt{2}}{2} < \frac{2}{2}$$

$f'(-1) = -$

$f'(0) = +$

$f'(1) = -$

f is pos, dec, & cont. for $x \geq 1$.

$$\int_1^{\infty} \frac{4x}{2x^2+1}\,dx$$

$u = 2x^2+1$

$du = 4x\,dx$

$$\int_1^{\infty} \frac{1}{u}\,du$$

$$\ln(2x^2+1)\Big|_1^{\infty}$$

$F(\infty) - F(1)$

$\ln(\infty) - \ln(3)$

$= \infty \Rightarrow$ diverges

Since $\int_1^{\infty} \frac{4x}{2x^2+1}\,dx$ diverges, $\sum_{n=1}^{\infty} \frac{4n}{2n^2+1}$ also diverges by the Integral Test.

⑮ $\sum_{n=1}^{\infty} n e^{-n}$

Let $f(x) = x e^{-x}$

$f'(x) = (x)(-e^{-x}) + (e^{-x})(1)$

$= -e^{-x}(x-1)$

$f' \quad + \underset{1}{|} -$

f is pos, dec, $ cont. for $x \geq 1$

$\int_1^{\infty} x e^{-x}\, dx$

$u = x$

$du = 1dx$

$v = -e^{-x}$

$dv = e^{-x}$

$uv - \int v\, du$ (Int. by Parts Formula)

$-xe^{-x}\Big|_1^{\infty} + \int_1^{\infty} +e^{-x}\, du$

$-xe^{-x}\Big|_1^{\infty} - e^{-x}\Big|_1^{\infty}$

$\frac{-x}{e^x} - \frac{1}{e^x}\Big|_1^{\infty}$

$\frac{-x-1}{e^x}\Big|_1^{\infty}$

$F(\infty) - F(1)$

$\frac{-\infty}{\infty} + \left(\frac{+2}{e}\right)$

L'Hopital's Rule Applies

$\frac{-1}{e^x}\Big|_1^{\infty} + \frac{2}{e}$

$\left[\frac{-1}{\infty} - \frac{1}{e}\right] + \frac{2}{e} = 0 + \frac{1}{e} = \frac{1}{e}$

Since $\int_1^{\infty} x e^{-x}\, dx$ converges, $\sum_{n=1}^{\infty} ne^{-n}$ also converges by the Integral Test.

⑯ $1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \ldots$

$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$

$\sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$

$p = 1/2 < 1$

Divergent (p-series)

⑰ $\sum_{n=1}^{\infty} \frac{1}{n^{\pi}}$

$p = \pi > 1$

convergent (p-series)

⑱ $\sum_{n=1}^{\infty} \frac{1}{\sqrt[5]{n}}$

$\sum_{n=1}^{\infty} \frac{1}{n^{1/5}}$

$p = 1/5 < 1$

Divergent (p-series)

⑲ $\sum_{n=1}^{\infty} \frac{n^2}{e^n}$

Let $f(x) = \frac{x^2}{e^x}$

$f'(x) = \frac{(e^x)(2x) - (x^2)(e^x)}{(e^x)^2} = \frac{x\cancel{e^x}(2-x)}{\cancel{e^x} \cdot e^x} = \frac{x(2-x)}{e^x}$

C.N:

$x = 0, 2$

$f' \quad - \underset{0}{|} + \underset{2}{|} -$

Note: f is pos, cont, $ dec. on $(2, \infty)$

$\sum_{n=1}^{\infty} \frac{n^2}{e^n} = \frac{1}{e} + \sum_{n=2}^{\infty} \frac{n^2}{e^n}$

$\int_2^{\infty} \frac{x^2}{e^x}\, dx = \int_2^{\infty} x^2 e^{-x}\, dx$

$u = x^2$

$du = 2x dx$

$v = -e^{-x}$

$dv = e^{-x}$

$-x^2 e^{-x} + \int_2^{\infty} +e^{-x} \cdot 2x\, dx$

$u = 2x$

$du = 2dx$

$v = -e^{-x}$

$dv = e^{-x}$

$-x^2 e^{-x} - 2xe^{-x} + \int_2^{\infty} +2e^{-x}\, dx$

$-x^2 e^{-x} - 2xe^{-x} - 2e^{-x}\Big|_2^{\infty}$

$(0 - 0 - 0) - \left(\frac{-4}{e^2} - \frac{4}{e^2} - \frac{2}{e^2}\right)$

$\Rightarrow 1/e + 10/e^2 \Rightarrow$ converges $\therefore$ $1/e + \sum_{n=2}^{\infty} n^2/e^n$ also converges by the Integral Test.

⑳ $\sum_{n=0}^{\infty} 3\cdot\left(\frac{5}{9}\right)^n$ | $3\sum_{n=0}^{\infty}\left(\frac{5}{9}\right)^n$ | convergent (Geometric Series w/ $r = 5/9 < 1$)

㉑ $\sum_{n=1}^{\infty} \frac{1}{n^{1.04}}$ | convergent (p-series w/ $p = 1.04 > 1$)

㉒ $\sum_{n=1}^{\infty} \frac{2}{n^3} - \frac{2}{n^4}$ | $\sum_{n=1}^{\infty}\frac{2}{n^3} - \sum_{n=1}^{\infty}\frac{2}{n^4}$ | $2\sum_{n=1}^{\infty}\frac{1}{n^3} - 2\sum_{n=1}^{\infty}\frac{1}{n^4}$

convergent (p-series w/ $p = 3 > 1$) →

↑ convergent (p-series w/ $p = 4 > 1$)

$\therefore \sum_{n=1}^{\infty}\frac{2}{n^3} - \frac{2}{n^4}$ converges (sum of two convergent p-series)

㉓ $\sum_{n=1}^{\infty}\frac{3}{n} - \frac{3}{n+1}$ | $\left(\frac{3}{1} - \frac{\cancel{3}}{\cancel{2}}\right) + \left(\frac{\cancel{3}}{\cancel{2}} - \frac{\cancel{3}}{\cancel{3}}\right) + \dots + \left(\frac{\cancel{3}}{\cancel{n-1}} - \frac{\cancel{3}}{\cancel{n}}\right) + \left(\frac{\cancel{3}}{\cancel{n}} - \frac{3}{n+1}\right)$

$= \lim_{n\to\infty} 3 - \frac{3}{n+1} = 3 - \frac{3}{\infty} = 3 - 0 = 3$ | Series converges to 3 using the telescoping series.

㉔ $\frac{\ln(2)}{2} + \frac{\ln(3)}{3} + \frac{\ln(4)}{4} + \dots$ | $\sum_{n=2}^{\infty}\frac{\ln(n)}{n}$ | Let $f(x) = \frac{\ln(x)}{x}$

$f'(x) = \frac{(x)(1/x) - (\ln x)(1)}{x^2} = \frac{1-\ln x}{x^2}$ | C.N: $x = e$
 P.o.I; $x = 0$ | f' $\underset{0}{\vdots}\ +\ \underset{e}{|}\ -$ | $f'(e^2) = \frac{1-2}{e^4} = -$

f is pos, dec, \$ cont. for $x > e$ | $\sum_{n=2}^{\infty}\frac{\ln(n)}{n} = \frac{\ln(2)}{2} + \sum_{n=3}^{\infty}\frac{\ln(n)}{n}$ | $\int_3^{\infty}\frac{\ln(x)}{x}\,dx$ | $u = \ln(x)$
 $du = 1/x\,dx$

$\int_3^{\infty} u\,du$ | $\left.\frac{u^2}{2}\right|_3^{\infty}$ | $\left.\frac{(\ln x)^2}{2}\right|_3^{\infty}$ | $F(\infty) - F(3)$ | $\left(\frac{(\ln\infty)^2}{2}\right) - \left(\frac{(\ln 3)^2}{2}\right)$ | $\infty - \# = \infty$

Since $\int_3^{\infty}\frac{\ln(x)}{x}\,dx$ diverges, $\sum_{n=3}^{\infty}\frac{\ln(n)}{n}$ also diverges by the Integral Test. And since $\sum_{n=3}^{\infty}\frac{\ln(n)}{n}$ diverges $\Rightarrow \frac{\ln(2)}{2} + \sum_{n=3}^{\infty}\frac{\ln(n)}{n}$ also diverges.